Suppression of deterministic diffusion by noise

Peter Reimann
Limburgs Universitair Centrum, B-3590 Diepenbeek, Belgium
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A one-dimensional discrete-time dynamics near the onset of deterministic diffusion additively coupled to weak Gaussian white noise is studied. The diffusion coefficient is related to the escape rate out of an interval and a method for the calculation of this rate is introduced which, unlike previous investigations, fully takes into account all relevant escape paths. It is found that the diffusion coefficient of the deterministic dynamics may be considerably reduced by the noise in agreement with numerical simulations. It is shown that certain fundamental problems in the concept of deterministic diffusion without noise do not affect the validity of our result.

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I. INTRODUCTION

Deterministic diffusion refers to the asymptotically linear growth of the mean square displacement in a purely deterministic system. A well studied situation is that of a one-dimensional dynamics

$$x_{n+1}=f_\mu(x_n) \tag{1}$$

in discrete time n [1]. Under certain circumstances, one finds that there exists a nonvanishing finite diffusion coefficient

$$D:=\lim_{n\to\infty}\frac{\langle x_n^2\rangle}{2n}, \tag{2}$$

where the average $\langle\ \rangle$ is taken over a smooth distribution of initial positions x_0 with bounded support. The dynamics (1) is assumed to be invariant under inversion and discrete translation, i.e., $f_\mu(-x)=-f_\mu(x)$ and $f_\mu(l+x)=l+f_\mu(x)$, $l\in\mathbb{Z}$. Moreover, the map $f_\mu(x)$ has one local maximum and minimum of order $z>0$ per unit cell $[l,l+1]$ with unstable fixed points at the integers. It depends smoothly on the parameter μ and the argument x except for possible nondifferentiabilities at the local extrema. A common example belonging to the class $z=2$ is the climbing-sine map

$$f_\mu(x)=x+\mu\sin(2\pi x)\ ; \tag{3}$$

see Fig. 1.

As pointed out in [2,3], deterministic diffusion only appears for those parameter values μ that lead to fully developed chaos in the reduced map,

$$\tilde f_\mu:\ [0,1]\to[0,1],\quad \tilde f_\mu(x):=f_\mu(x)\mathrm{mod}1\ , \tag{4}$$

i.e., for μ values such that $\tilde f_\mu(x)$ has a strange attractor covering the whole unit interval [0,1]. For example, if we denote by

$$\epsilon:=f_\mu(x^*) \tag{5}$$

the distance between the upper boundary $y=0$ of the unit cell $[-1,0]$ and the value of $f_\mu(x)$ at the local maximum $x=x^*\in[-1,0]$, see Fig. 1, then for $\epsilon\le 0$ the function $f_\mu(x)$ maps every unit cell $[l,l+1]$ into itself and consequently the diffusion coefficient (2) vanishes ($\epsilon=0$ is an exception showing fully developed chaos of the reduced map but no deterministic diffusion). More generally, for $z>1$ there is an infinity of parameter intervals belonging to periodic windows of the reduced map (4) where the diffusion coefficient either vanishes or diverges [2,3]. The way in which the diffusion coefficient goes to zero or infinity as one approaches such a parameter interval has been studied in detail [1–5].

In this paper we consider the dynamics (1) additively coupled to Gaussian white noise

$$x_{n+1}=f_\mu(x_n)+\xi_n,\quad P(\xi_n)=(2\pi\sigma^2)^{-1/2}e^{-\xi_n^2/2\sigma^2} \tag{6}$$

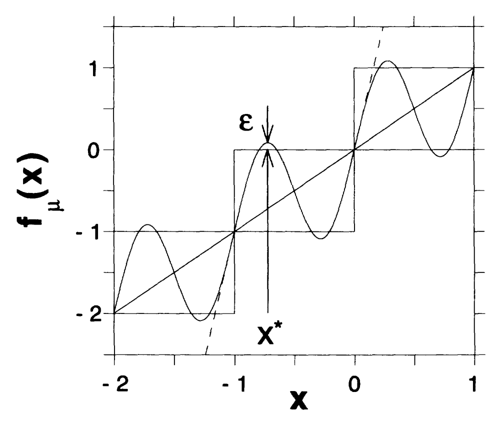

FIG. 1. The solid curve represents the climbing-sine map (3) with $\mu=0.82$ for x values out of the three unit cells $[-2,-1]$, $[-1,0]$, and $[0,1]$. The local maximum within the unit cell $[-1,0]$ is denoted by x^* and the quantity $\epsilon=f_\mu(x^*)$ is introduced in (5). The critical value of μ for which ϵ vanishes and thus every unit cell is mapped onto itself is given by $\mu=0.732\ldots$. The dashed curve is the map considered in Sec. II for the calculation of the escape rate out of $[-1,0]$ which coincides with the original $f_\mu(x)$ on the unit cell $[-1,0]$ and is linear outside; see Eq. (10).

of small noise strength σ. The inclusion of a small amount of noise is not only suggested from the experimental point of view, but also avoids some theoretical difficulties of the deterministic dynamics (1), as will be pointed out in Sec. IV.

We will study the diffusion coefficient (2) for the noisy dynamics (6) near the upper boundary of a periodic window of the reduced map (4) where the diffusion coefficient vanishes in the absence of noise. For the sake of convenience we confine ourselves to the simplest case characterized by small values of the parameter ϵ in (5). It might seem that this problem has already been solved in the seminal paper by Geisel and Nierwetberg [1]. However, for nonvanishing noise strengths σ their approach is only valid for maps $f_\mu(x)$ with discontinuities instead of unstable fixed points at the integers (see condition (3c) in [1]).

We proceed as follows. In Sec. II the diffusion coefficient is identified with half the escape rate out of a unit cell $[l,l+1]$ and a method for the calculation of this rate is introduced which overcomes the shortcomings or limitations of previous investigations [1,6–9] of the problem. Our main result is the scaling law (28) or equivalently (30) for the diffusion coefficient. The discussion of this result and the comparison with numerical simulations is carried out in Sec. III which can be understood without having followed the calculations in Sec. II. It is found that the diffusion coefficient of the deterministic dynamics (1) may be considerably reduced in the presence of noise (6) and an intuitive explanation of this effect is given for a special case. In Sec. IV some fundamental problems in the theory of purely deterministic diffusion are mentioned which were first pointed out by Fujisaka and Grossmann [3]. Then, with these problems in mind, the range of validity of our result is explored. In Sec. V the results are summarized and the properties of the diffusion coefficient for general parameter values μ but still small noise strengths σ are briefly outlined.

II. CALCULATION OF THE DIFFUSION COEFFICIENT

We consider the Langevin equation (6) for small noise strengths σ close to the onset of deterministic diffusion, i.e., for small parameter values ϵ in (5). We first derive a relation between the diffusion coefficient (2) and the escape rate k out of a unit cell $[l,l+1]$. To this end let us focus on the unit cell $I:=[-1,0]$. Furthermore, we introduce a small "boundary region"

$$B:=[-b,b], \quad b:=O(\sigma/u)\ , \tag{7}$$

$$u:=f'_\mu(0)>1 \tag{8}$$

of the unstable fixed point at $x=0$ and an "interior" $\hat{I}:=[-1+b,-b]$ of the cell I. The essence of these definitions is that within $\hat{I}$ the deterministic map plays the dominant role in the Langevin equation (6), i.e., a typical trajectory looks qualitatively the same as in the absence of noise. On the other hand, within B the noise term dominates the dynamics, yielding essentially Brownian motion.

As in the noiseless case [2,3,10], a typical realization of the Langevin equation (6) which starts within $\hat{I}$ will irregularly bounce around in $\hat{I}$ for a long time. The usual way the realization finally leaves $\hat{I}$, for concreteness say towards the right hand side, is after having visited a small neighborhood of the local maximum x^* of the map $f_\mu(x)$ within I, see Fig. 1. If the realization directly jumps into the "interior" $\hat{J}:=[b,1-b]$ of the adjacent unit cell [0,1] it typically will stay there for another long period of time and thus counts as a "successfull escape attempt." On the other hand, if it enters the boundary region B it will follow essentially a Brownian motion within B for a period of time which typically is much shorter than the one spent in $\hat{I}$. Then, possibly after having crossed and recrossed the boundary $x=0$ of the unit cell I several times, it enters into $\hat{J}$ or returns into $\hat{I}$ for another long period of time and thus counts as a "successful" or "unsuccessful escape attempt," respectively.

The probability of a successful escape attempt per time step determines the escape rate k (note that in this paper we take the convention that the rate accounts for escapes across both the upper and lower boundary of a unit cell). Clearly, this rate k is the same for any unit cell $[l,l+1]$. Since the typical time between successive escapes is very large we further assume that they are uncorrelated (the validity of this assumption is discussed in more detail in Sec. IV, see also Ref. [3]). In a coarse grained description we are thus dealing with a Markovian nearest-neighbor random walk on the one-dimensional lattice $\mathbb{Z}$ with a diffusion coefficient (2) given by [11]

$$D=\frac{k}{2}\ . \tag{9}$$

Next we address the calculation of the escape rate k out of the unit cell $I=[-1,0]$. In previous investigations of this problem [6,7] the escape rate has been determined from those realizations of the Langevin equation (6) which directly leave the unit cell I after having visited a small neighborhood of the local maximum x^* of the map $f_\mu(x)$. In other words, two important effects are neglected [12]: (1) realizations may first visit $[-b,0]$ before escaping, and (2) realizations may return from $[0,b]$ into $\hat{I}$ after a short time. However, these effects are truly negligible only for $b=0$, i.e., for vanishing noise $\sigma=0$ or infinite slope $u\to\infty$, see (7). For finite σ and u the rate is enhanced by the first and reduced by the second effect in comparison with $\sigma=0$ or $u\to\infty$ and it is not clear which of these competing contributions will dominate. In particular, our central result that the rate and thus the diffusion coefficient can be smaller in the presence rather than in the absence of noise has no simple intuitive explanation for finite u.

In the limit $u\to\infty$ the map $f_\mu(x)$ has discontinuities instead of unstable fixed points at the integers. Due to the relation (9) between the diffusion coefficient and the escape rate the calculations of Geisel and Nierwetberg [1] are thus essentially equivalent to those in the above mentioned Refs. [6,7]. First order corrections for finite u have recently been derived in [8] by methods which to some extent are similar to ours. We finally mention that in the regime $-1\ll\epsilon\ll-\sigma$ the correct exponentially leading Arrhenius factor of the rate has been calculated in [9], but without the nonexponential prefactor, see also the discussion below Eq. (33).

In the remainder of this section we outline the main steps of our method for calculating the escape rate k. More details and generalizations will be given elsewhere [13]. In order to determine the escape rate k out of the unit cell $I=[-1,0]$ for weak noise σ and small parameters ϵ, for $x\geq 0$ the map $f_\mu(x)$ can be replaced by its linearization about the unstable fixed point at $x=0$:

$$f_\mu(x)=ux \tag{10}$$

and similarly for $x\leq -1$, see Fig. 1. Let $W_0(x)$ be an initial distribution of realizations of the Langevin equation (6) which is mainly concentrated in I. Then the evolution of the probability distributions $W_n(x)$ is governed by the master equation

$$W_{n+1}(x)=\int_{-\infty}^{\infty} dy\, P(x|y)W_n(y)\ , \tag{11}$$

where $P(x|y)$ is the transition probability to go in one time step from y to x. According to (6) we have

$$P(x|y)=\int_{-\infty}^{\infty} d\xi\, \delta[x-f_\mu(y)-\xi]P(\xi) = \frac{\exp\{-[x-f_\mu(y)]^2/2\sigma^2\}}{\sqrt{2\pi\sigma^2}}\ . \tag{12}$$

Let us introduce a time-dependent decay rate $k(n)$ which is given by the relative decrease of the population in the interval I after one time step:

$$k(n):=\frac{\int_I dx\, W_n(x)-\int_I dx\, W_{n+1}(x)}{\int_I dx\, W_n(x)}\ . \tag{13}$$

For large times n the rate $k(n)$ converges towards an asymptotic value which evidently is the escape rate k in which we are interested [12]. The criterion for this convergence is the condition that the system must be close to the quasi-invariant state, i.e., $W_{n+1}(x)=W_n(x)$ must be fulfilled in good approximation. Clearly, by an appropriate choice of the initial distribution $W_0(x)=:W(x)$, obeying

$$W(x)=\int_{-\infty}^{\infty} dy\, P(x|y)W(y) \tag{14}$$

in good approximation, one has $k(n)\simeq k$ already for $n=0$. Then, making use of probability conservation $\int_{-\infty}^{\infty}dx\, W_{n+1}(x)=\int_{-\infty}^{\infty}dx\, W_n(x)$, which follows from (11) and (12), the rate (13) can be rewritten

$$k=\frac{\int_{\bar{I}}dx\int_{-\infty}^{\infty} dy\, P(x|y)W(y)-\int_{\bar{I}}dx\, W(x)}{\int_I dx\, W(x)}\ , \tag{15}$$

where $\bar{I}:=\mathbb{R}\setminus I$ is the complement of I. If one further requires that the quasi-invariant density $W(x)$ is symmetric about the center $x=-\frac{1}{2}$ of the interval I, i.e., $W(-\frac{1}{2}-x)=W(-\frac{1}{2}+x)$, it can be readily shown that (15) is equivalent to

$$k=2\frac{\int_0^{\infty}dx\left[\int_{-\infty}^{\infty} P(x|y)W(y)dy-W(x)\right]}{\int_{-1}^{0}dx\, W(x)}\ . \tag{16}$$

Next the quasi-invariant density $W(x)$ respecting symmetry about $x=-\frac{1}{2}$ and approximately fulfilling the master equation (14) is determined. To this end we first consider the "true" invariant density $\rho(x)$ describing the stationary probability distribution on the interval I in the absence of noise $\sigma=0$ at fully developed chaos $\epsilon=0$. The results of Refs. [14–16] suggest that $\rho(x)$ typically behaves as

$$\rho(x)\sim|x_b-x|^{1/z-1} \tag{17}$$

close to a boundary x_b of the interval I and is positive, bounded, and smooth on the remainder of I, see also the example discussed at the end of Sec. III. Thus it is plausible to assume that sufficiently small changes in σ and ϵ will mainly affect the singularities (17) near the boundaries of I, whereas one has in good approximation

$$W(x)=\rho(x) \tag{18}$$

on the remainder of I. A more detailed discussion of the approximation (18) will be given in [13], see also Sec. IV.

For symmetry reasons and due to (18) we are left to determine $W(x)$ for $x\geq x_{\text{th}}$, where the threshold x_{th} is situated slightly below the upper boundary $x=0$ of I. More precisely, $x_{\text{th}}<0$ is chosen such that the linearization (10) of $f_\mu(x)$ applies in good approximation for all $x\geq x_{\text{th}}$. Since this choice is independent of σ and ϵ we can assume $|x_{\text{th}}|$ to be much larger than σ and $|\epsilon|$ in the sequel.

In order to determine $W(x)$ for $x\geq x_{\text{th}}$ from the master equation (14) we recast the right hand side into an appropriate form. For $x\geq x_{\text{th}}$ the only y values for which the transition probability $P(x|y)$ in (12) notably contributes are contained in the domain $y\geq x_{\text{th}}$ and in a small neighborhood $[x^*-\delta,x^*+\delta]$ about the local maximum x^* of $f_\mu(x)$ within I, see Fig. 1. The same is true also if $P(x|y)$ is multiplied by $W(y)$ as can be concluded from (18) and the behavior of $\rho(x)$ mentioned below (17). Due to our choice of x_{th} the map of $f_\mu(y)$ in the transition probability (12) can be approximated by the linearization (10) for $y\geq x_{\text{th}}$. Similarly to x_{th}, the quantity δ can be chosen much larger than σ and $|\epsilon|$ but still sufficiently small such that within a δ neighborhood of the local maximum $x^*\in I$ the map $f_\mu(x)$ is well approximated by

$$f_\mu(x^*+\Delta x)=\epsilon-a|\Delta x|^z, \tag{19}$$

where $a>0$, z is the order of the local extrema, and ϵ is introduced in (5). Collecting everything, for $x\geq x_{\text{th}}$ the right hand side of the master equation (14) can be rewritten

$$\int_{-\infty}^{\infty}dy\, P(x|y)W(y)=\int_{-\delta}^{\delta}\frac{dy}{\sqrt{2\pi\sigma^2}}\rho(x^*+y)\exp\left\{-\frac{[x-\epsilon+a|y|^z]^2}{2\sigma^2}\right\}+\int_{x_{\text{th}}}^{\infty}\frac{dy}{\sqrt{2\pi\sigma^2}}W(y)\exp\left\{-\frac{[x-uy]^2}{2\sigma^2}\right\}\ . \tag{20}$$

Since $\rho(x)$ is smooth near $x=x^*$, see below (17), we can replace $\rho(x^*+y)$ in (20) by $\rho(x^*)$. Next we note that our assumptions so far on the quantities $x_{\rm th}$ and δ still allow a choice which fulfills the relation $f(x^*\pm\delta)\simeq\epsilon-a|\delta|^z<x_{\rm th}$. Then, the exponential term in the first integral on the right-hand side of (20) takes its maximum with respect to y in the interior of $[-\delta,\delta]$ for any $x\geq x_{\rm th}$. Thus for sufficiently small σ and ϵ the integration domain $[-\delta,\delta]$ can be extended over the whole real axis. Similarly, for any $x\geq x_{\rm th}$ the exponential term in the second integral on the right-hand side of (20) takes its maximum with respect to y in the interior of the integration domain $[x_{\rm th},\infty]$ and, in particular, is negligible for y values even below $x_{\rm th}$. We assume that the same stays true if the exponential term is multiplied by $W(y)$ which will turn out to be self-consistent with our final result for $W(y)$. Hence the integration limit $x_{\rm th}$ can be replaced by $-\infty$. In summary, we arrive at

$$\int_{-\infty}^{\infty}dy\,P(x|y)W(y)=w_0(x)+\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi\sigma^2}}W(y)\exp\left\{-\frac{[x-uy]^2}{2\sigma^2}\right\} \tag{21}$$

for $x\geq x_{\rm th}$, where we introduced

$$w_0(x):=\rho(x^*)\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{[x-\epsilon+a|y|^z]^2}{2\sigma^2}\right\}. \tag{22}$$

Equation (21) suggests the ansatz

$$W(x)=\sum_{i=0}^{m-1}w_i(x) \tag{23}$$

for $x\geq x_{\rm th}$, where the $w_i(x)$ are defined by the recursion relation

$$w_{i+1}(x):=\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi\sigma^2}}w_i(y)\exp\left\{-\frac{[x-uy]^2}{2\sigma^2}\right\} \tag{24}$$

and the initial condition (22). Note that the functions $w_i(x)$ are defined on the whole real axis by (22) and (24). The number m of summands in (23) remains to be determined. From (21), (23), and (24) it follows that

$$\int_{-\infty}^{\infty}P(x|y)W(y)dy-W(x)=w_m(x) \tag{25}$$

for $x\geq x_{\rm th}$. The evaluation of $w_m(x)$ from (22) and (24) is straightforward, reading

$$w_m(x)=\rho(x^*)\int_{-\infty}^{\infty}\frac{dy}{\sqrt{\pi s_m}}\times\exp\left\{-\frac{[x+u^m(a|y|^z-\epsilon)]^2}{s_m}\right\}, \tag{26}$$

where we introduced $s_m:=2\sigma^2[u^{2m+2}-1]/[u^2-1]$. It can be readily seen that this implies $w_m(x)\to 0$ for $m\to\infty$. In view of (25) it follows that our ansatz (23) indeed fulfills the master equation (14) for $x\geq x_{\rm th}$ in good approximation if m is chosen sufficiently large.

Due to (25) the numerator in the escape rate (16) is given by $\int_0^\infty dx\,w_m(x)$. As for the denominator, we assume that $\rho(x)$ is normalized on the interval $I=[-1,0]$ and that the differences between $\rho(x)$ and $W(x)$ in the small boundary regions of I do not change this normalization in good approximation. This implies

$$k=2\int_0^\infty dx\,w_m(x) \tag{27}$$

for sufficiently large values of m. Inserting (26) into (27) shows that there is indeed a well-defined limit $m\to\infty$ and after some straightforward manipulations one arrives at the central formula for the diffusion coefficient (9):

$$D=\left[\left[\frac{2u^2}{u^2-1}\right]^{1/2}\frac{\sigma}{a}\right]^{1/z}F\left[\left[\frac{u^2-1}{2u^2}\right]^{1/2}\frac{\epsilon}{\sigma}\right]\rho(x^*), \tag{28}$$

$$F(x):=\int_0^\infty dy\,\mathrm{erfc}\,(y^z-x), \tag{29}$$

where $\mathrm{erfc}(x):=2\pi^{-1/2}\int_x^\infty e^{-y^2}dy$ is the complementary error function.

III. DISCUSSION OF THE DIFFUSION COEFFICIENT

The diffusion coefficient (28) has the form of a scaling law where the scaling function $F(x)$ is given by (29). The specific properties of the map $f_\mu(x)$ in the noisy dynamics (6) only enter into the diffusion coefficient (28) through the slope u at the unstable fixed points (8), the quantities z, ϵ, and a describing the local extrema (19), and the invariant density $\rho(x^*)$ for $\epsilon=\sigma=0$ at the local extrema of the map. The scaling function (29) is universal for fixed order z of the local extrema. Moreover, $F(x)$ is a monotonically increasing function of x, approaching zero and infinity for $x\to-\infty$ and $x\to+\infty$, respectively. In the limit $u\to\infty$, corresponding to discontinuities instead of unstable fixed points, the result of Geisel and Nierwetberg [1] is recovered.

In order to discuss the behavior of the diffusion coefficient for fixed parameter ϵ as a function of the noise strength σ it is convenient to recast (28), (29) into the form

$$D=\left[\frac{|\epsilon|}{a}\right]^{1/z}G\left[\left[\frac{2u^2}{u^2-1}\right]^{1/2}\frac{\sigma}{\epsilon}\right]\rho(x^*), \tag{30}$$

$$G(x):=\int_0^\infty dy\,\mathrm{erfc}\left[\frac{y^z-\mathrm{sgn}(x)}{|x|}\right], \tag{31}$$

where $\mathrm{sgn}(x):=x/|x|$ is the signum function. For x approaching zero from above one finds from (31) that $G(x\downarrow 0)=2$ since $\mathrm{erfc}(+\infty)=0$ and $\mathrm{erfc}(-\infty)=2$. Thus for positive ϵ values the diffusion coefficient (30) approaches the correct σ-independent behavior [1–3]

$$D=2\left[\frac{|\epsilon|}{a}\right]^{1/z}\rho(x^*) \tag{32}$$

in the deterministic limit $\sigma \ll \epsilon$. For x approaching zero from below one can make use in (31) of the asymptotic property $\mathrm{erfc}(y) \simeq e^{-y^2}/[\sqrt{\pi}y]$ for large y. After some calculations one finds for the diffusion coefficient (30) that

$$D = \frac{\Gamma(1/z+1)}{2^{1/z}\sqrt{\pi}}\rho(x^*)\left[\frac{|\epsilon|}{a}\right]^{1/z} \times \left[\left[\frac{2u^2}{u^2-1}\right]^{1/2}\frac{\sigma}{|\epsilon|}\right]^{2/z+1}\exp\left\{-\frac{u^2-1}{2u^2}\frac{\epsilon^2}{\sigma^2}\right\} \tag{33}$$

for $\epsilon<0$ and $\sigma \ll |\epsilon|$. Equation (33) has the form of an Arrhenius law as has to be expected from the relation (9) between the diffusion coefficient and the escape rate k out of a unit cell. In particular, Eqs. (30) and (33) yield $G(x\uparrow 0)=0$ corresponding to the fact that for $\epsilon<0$ the diffusion coefficient vanishes in the absence of noise. Note that the noise strength σ can take only non-negative values. Thus the discontinuities of the scaling function (31) at $x=0$ and of the diffusion coefficient (30) at $\sigma=0$ are of no physical relevance. The exponentially leading Arrhenius factor of the rate following from (9) and (33) has previously been derived in [9].

In the remainder of this section we restrict ourselves to positive values of ϵ and hence to non-negative x in (31). It can be shown analytically that $G(x)$ in (31) takes a global minimum at a positive x value $x_{\min}(z)$ provided $z>1$, but for convenience we refer to the numerical evaluation of (31) plotted in Fig. 2. Thus for a fixed positive ϵ the diffusion coefficient as a function of σ takes a global minimum at $\sigma_{\min}=\epsilon x_{\min}(z)\sqrt{(u^2-1)/(2u^2)}$ independent of a. In other words, deterministic diffusion is suppressed by the noise. The position $x_{\min}(z)$ of the minimum as well as the maximal reduction

$$R(z) := \frac{G(x\downarrow 0)-G(x_{\min}(z))}{G(x\downarrow 0)} \tag{34}$$

of the diffusion coefficient monotonically increase with z, see Figs. 3 and 4. One can show analytically that for large z the minimum $x_{\min}(z)$ approaches $2z/\sqrt{\pi}$ and the reduction of the diffusion coefficient

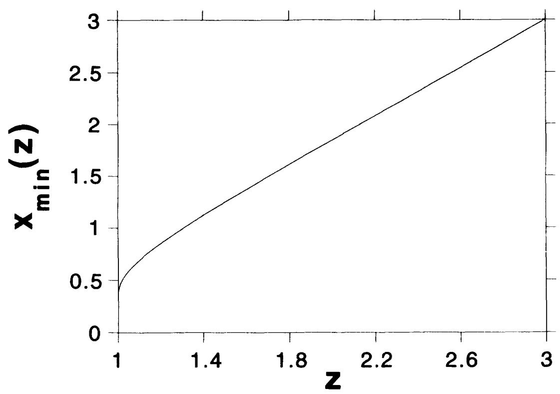

FIG. 3. The minimum $x_{\min}(z)$ of the scaling function $G(x)$ in (31) as a function of z for $z>1$.

$R(z)$ converges towards $\frac{1}{2}$. For $z\to 1$ one finds that $x_{\min}(z)\sim|\ln(z-1)|^{-1/2}$ and $R(z)\sim(z-1)/|\ln(z-1)|$. The logarithmic terms are due to the fact that all derivatives of $G(x)$ vanish for $x\downarrow 0$ and $z>1$.

In order to compare the diffusion coefficient (30) with numerical simulations let us introduce for the sake of convenience a shift of the origin by $\frac{1}{2}$. For the map $f_\mu(x)$ we take the example belonging to the class $z=2$,

$$f_\mu(x) := 4\mu x^3-(\mu-1)x \tag{35}$$

for $x\in[-\frac{1}{2},\frac{1}{2}]$ and $f_\mu(l+x):=l+f_\mu(x)$, $l\in\mathbf{Z}$, otherwise. The fact that $f_\mu(x)$ has discontinuous second derivatives at $x=l+\frac{1}{2}$ does not play a role. The following properties of $f_\mu(x)$ are straightforward:

$$\epsilon = \frac{\mu-1}{3}\left[\frac{\mu-1}{3\mu}\right]^{1/2}-\frac{1}{2},\quad x^*=-\frac{1}{4},\quad a=12,\quad u=9\,. \tag{36}$$

For $\mu=4$ the parameter ϵ vanishes and $f_{\mu=4}(x)=T_3(2x)/2$ on $[-\frac{1}{2},\frac{1}{2}]$, where $T_3(x)$ is the cubic

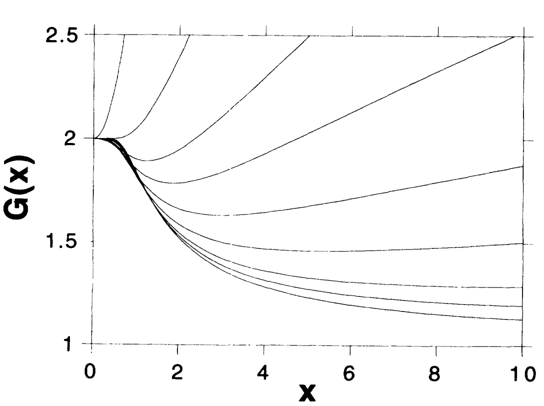

FIG. 2. The universal scaling function $G(x)$ in (31) for $z=0.5$, 1, 1.5, 2, 3, 5, 10, 20, and 100 (from above). The minima $x_{\min}(z)$ of $G(x)$ for $z=10$, 20, and 10 are beyond $x=10$.

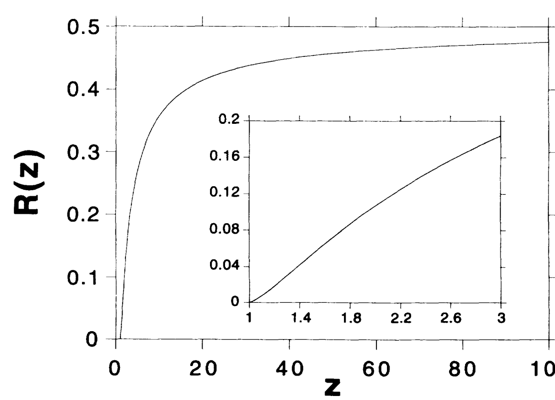

FIG. 4. The maximal reduction $R(z)$ of the diffusion coefficient defined in (34) as a function of z for $z>1$. The inset shows the behavior near $z=1$ in more detail.

Chebyshev polynomial. Due to this fact the invariant density $\rho(x)$ for $\epsilon=\sigma=0$ on $[-\frac{1}{2},\frac{1}{2}]$ is known to be $\rho(x)=2/[\pi\sqrt{1-4x^2}]$ [16], yielding

$$\rho(x^*)=\frac{4}{\pi\sqrt{3}} \ . \tag{37}$$

Hence the diffusion coefficient in (30) is completely fixed. The comparison with numerical simulations is shown in Fig. 5.

The reduction of the diffusion coefficient by noise is equivalent to a similar behavior of the escape rate k out of a unit cell according to (9), which has recently been observed also in a numerical study by Franaszek [17]. As seen at the beginning of Sec. II, a simple intuitive explanation of this effect cannot be given for finite values of u. However, in the limit $u\to\infty$ the essential mechanism which leads to a noise-induced reduction of the escape rate (for $z>1$) can be easily understood. For concreteness we consider the escapes from the unit cell $[-1,0]$ into $[0,1]$. Without noise, such an escape takes place upon iteration of the deterministic dynamics (1) if and only if a realization visits an $(\epsilon/a)^{1/z}$ neighborhood of the local maximum x^* of $f_\mu(x)$ within $[-1,0]$, cf. (19) and Fig. 1. In the presence of noise some additional realizations slightly outside the $(\epsilon/a)^{1/z}$ neighborhood of x^* will escape, but on the other hand a part of the realizations within this neighborhood will not escape. Since near x^* the probability density $W(x)$ is approximately constant, see (18) and below (17), and the map $f_\mu(x)$ has negative curvature for $z>1$, it is obvious that the net effect of the noise will be a reduction of the escape rate for not too large noise strengths σ. This argument makes it also plausible that the reduction, e.g., for $z=2$ will be maximal for $\sigma=O(\epsilon)$ and approach $\frac{1}{2}$ for $z\to\infty$ and $\sigma\gg\epsilon$.

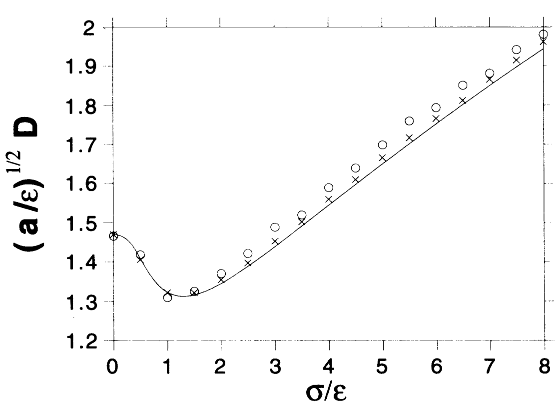

FIG. 5. Numerical simulations of the diffusion coefficient (2) for the noisy dynamics (6), (35) for $\epsilon=10^{-4}$ (circles) and $\epsilon=10^{-5}$ (crosses). The solid curve represents the theoretical result according to (30), (36), (37). The agreement between simulations and theory improves with decreasing values of ϵ and is very good for $\epsilon=10^{-5}$. The numerical uncertainty due to the finite number (50 000) of realizations and their finite length (2000) in (2) is below 1%. In particular, the numerical results were found to be practically independent of the length n in (2) for $n=100$ up to $n=10^4$.

IV. REMARKS ON THE VALIDITY OF THE RESULT

As a background for the following discussion we mention two unsolved problems in the theory of deterministic diffusion in the absence of noise: As pointed out in the Introduction, deterministic diffusion is ruled out for the entire μ intervals belonging to periodic windows of the reduced map (4), in particular for the small scale chaotic bands inside the windows [2] and the boundary points of the μ intervals [3,4]. Fully developed chaos of the reduced map and thus deterministic diffusion *may* occur only on the complement of these μ intervals. It is not known if the set of μ values where deterministic diffusion *does* occur has positive measure [3].

It has been shown [4] that the number of nondiffusive μ intervals has an infinity of accumulation points, e.g., for the climbing-sine map (3). In view of the similarity between the bifurcation diagrams of the climbing-sine map and the logistic map $g_\mu(x)=\mu x(1-x)$ we conjecture that the nondiffusive μ intervals are even dense on the μ axis for general maps $f_\mu(x)$ in (1) with $z>1$ [18]. Putting it differently, the diffusion coefficient is finite only on a set of isolated points μ, whereas it is zero or infinite on the μ intervals "in between" belonging to periodic windows of the reduced map. In fact, the situation reminds one of the erratic behavior of the Lyapunov exponent in the period doubling route to chaos [19]. In further analogy, both the diffusion coefficient and the Lyapunov exponent have been recognized to play the role of an order parameter close to certain "critical" values of the control parameter μ. In the period doubling route to chaos the corresponding asymptotic scaling laws do strictly speaking not apply to the Lyapunov exponent itself but rather to its envelope [19]. Similarly, in the case of deterministic diffusion without noise the asymptotic scaling laws derived in [1,2,4,5] are strictly speaking not valid for the diffusion coefficient itself. However, so far there seems to be no well-defined quantity to which these scaling laws apply from a rigorous point of view. The fact that the diffusion coefficient from numerical simulations seems to obey the scaling laws and to vanish or diverge only in the largest periodic windows of the reduced map can be understood as an artifact of extremely long transients: within a small periodic window the overwhelming part of initial points x_0 in (2) stays near the strange repeller of the map $f_\mu(x)$ for a very long time exhibiting transient chaos [10] which very closely resembles "true" fully developed chaos before the actual attracting set is approached. Besides this effect due to the finite number of time steps n in the numerical simulation of the diffusion coefficient (2) also the finite machine precision may play an important role [20] which, however, is beyond the scope of this paper.

Note that generically the above problems of deterministic diffusion without noise only arise in the case $z>1$, whereas for $z\le 1$ there are often no periodic windows of the reduced map except for the trivial one below $\epsilon=0$.

One may wonder why these problems did not arise anywhere in the derivation of the noisy diffusion coefficient (28), which in fact might seem to be valid even in the deterministic limit $\sigma \to 0$. However, in this derivation we made use of two rather poorly founded ingredients which in view of the above problems definitely need closer inspection.

We first address the approximation (18) made in the calculation of the escape rate. For small ϵ and σ it is not difficult to see that this approximation is always valid for $z \le 1$ and for $z > 1$ provided $\epsilon \ge 0$ since one remains in the regime of permanent or transient chaos of the map $f_\mu(x)$ (with linear continuations outside $[-1,0]$, see Eq. (10) and Fig. 1). In contrast to $z \le 1$, for $z > 1$ and negative ϵ there are periodic windows of the map $f_\mu(x)$ even arbitrarily close to $\epsilon = 0$. In this case it is obvious that $W(x)$ becomes very different from $\rho(x)$ for sufficiently small noise-strengths σ. However, one can prove [13] that for $z > 1$ a necessary and sufficient condition for (18) is given by

$$\epsilon \gg -\sigma^{(z-1)/z} . \tag{38}$$

Next we discuss the validity of the relation (9) between the escape rate and the diffusion coefficient. This relation relies on the assumption that successive escapes are uncorrelated, i.e., that there is a separation of time scales of the typical residence time in a unit cell and the relaxation to the quasistationary state after an escape [3]. Putting it differently, it is assumed that an ensemble of particles just escaping say from the unit cell $[-1,0]$ into $[0,1]$ approach a quasistationary distribution within $[0,1]$ much faster than the inverse escape rate and that the number of subsequent escapes into $[1,2]$ or back into $[-1,0]$ is negligible during this relaxation period.

Let us first consider the case that $\epsilon \ge O(\sigma)$. As discussed at the beginning of Sec. II, a typical realization escaping from $[-1,0]$ into $[0,1]$ visits a small neighborhood of the relative maximum x^* of $f_\mu(x)$ and arrives at $x \le O(\epsilon)$ at the next time step. If ϵ is not very much larger than σ, the realization may still recross the unstable fixed point at $x = 0$ with non-negligible probability, leading to a position $x = O(\sigma)$ after the last recrossing [21]. Thus an ensemble of escaping particles, each immediately after its last crossing of $x = 0$, is represented by a probability distribution centered at $x \le O(\epsilon)$ and of width equal or larger than the noise strength σ, where we made use of $\epsilon \ge O(\sigma)$. For concreteness we assume that the center is at $x = O(\epsilon)$ and the width is equal to $O(\sigma)$ which turns out to be the worst case. For our estimate at hand the linearization (10) of the map $f_\mu(x)$ about the unstable fixed point $x = 0$ can be used in order to describe the further evolution of the probability distribution. Neglecting for the moment the influence of the noise during this evolution, the position x_n of any realization after n time steps in given by $x_n = u^n x_0$. Thus the center of the probability distribution will be found at $x = O(u^n \epsilon)$ and the widths will be of order $O(u^n \sigma)$. In the presence of additive Gaussian noise (6), the position x_n of any realization has to be replaced by a Gaussian about x_n. Due to the linearization (10), the center of the distribution is still at $x = O(u^n \epsilon)$, while $O(u^n \sigma)$ is now a lower estimate for the width [22]. In the following we will use this lower estimate for the width, which again turns out to represent the worst case. Consequently, for small parameter values ϵ the center of the probability distribution leaves the boundary region about $x = 0$, i.e., becomes of order $O(1)$, within a time of order $O(\ln(1/\epsilon))$ and the width of the distribution becomes of order $O(\sigma/\epsilon)$ meanwhile [21].

Due to the definition of the ensemble, subsequent escapes into $[1,2]$ or back into $[-1,0]$ could safely be neglected when dealing within the boundary region about $x = 0$. After having left this region it becomes possible that the neighborhoods of the local extrema of $f_\mu(x)$ in $[0,1]$ are visited where the escape probability is no longer negligible. From (19) it follows that these neighborhoods of the local extrema are of order $O(\epsilon^{1/z})$ [21]. However, if the width of the probability distribution of order $O(\sigma/\epsilon)$ is much larger than these neighborhoods of order $O(\epsilon^{1/z})$ then the probability to visit these regions and hence to escape from $[0,1]$ is still negligible. We thus arrive at the condition

$$\epsilon \ll \sigma^{z/(z+1)} . \tag{39}$$

Under the action of the chaotic dynamics, the width of order $O(\sigma/\epsilon)$ becomes of order $O(1)$ after a time of order $O(\ln(\epsilon/\sigma))$. [We recall that $\epsilon \ge O(\sigma)$. Note that this does not exclude $\epsilon \le \sigma$, in which case we tactily set $O(\ln(\epsilon/\sigma))$ equal to zero.] Once the distribution is spread out over the whole interval $[0,1]$ the quasistationary state is typically reached within a few time steps [23]. Adding up the various contributions which have been described and estimated above, the total time from the escape out of $[-1,0]$ until quasistationarity within $[0,1]$ is reached is given by $O(\ln(1/\epsilon)) + O(\ln(\epsilon/\sigma)) + O(1)$. Since ϵ and σ are assumed to be small and $\epsilon \ge O(\sigma)$, the first term $O(\ln(1/\epsilon))$ dominates. As can be seen from (9) and the discussion of (30) in Sec. III, this term is negligible in comparison with $1/k$ for sufficiently small σ and ϵ values which fulfill $O(\sigma) \le \epsilon$ and (39). Furthermore, the escapes during this time can be neglected. Thus the separation of time scales is guaranteed.

In the case $\epsilon \le O(\sigma)$ and, in particular, for negative ϵ one can show by similar arguments that the separation of time scales is fulfilled without any additional condition on ϵ and σ. [The main difference to the case $\epsilon \ge O(\sigma)$ is that the center of the initial distribution is at $x = O(\sigma)$ instead of $x = O(\epsilon)$.] Hence for general ϵ and σ the separation of time scales is always guaranteed if they are sufficiently small and obey the condition (39). From the derivation of (39) it is can be concluded that this is a sufficient but not necessary condition.

To summarize, we found that (18) is valid for general small σ and ϵ if $z \le 1$ and under the additional necessary and sufficient condition (38) if $z > 1$. Further, the separation of time scales and thus the relation (9) between the diffusion coefficient and the escape rate holds under the sufficient condition (39). Consequently, under the same conditions also the results (28) or equivalently (30) for the diffusion coefficient are valid. In particular, the problems of deterministic diffusion without noise mentioned at the beginning of this section are avoided. We finally note that the above conditions still allow an arbitrary large range of ϵ/σ values in the diffusion coefficient (28) or (30) for sufficiently small σ.

V. SUMMARY AND OUTLOOK

We studied the noisy dynamics (6) for small noise strengths σ and small distances ϵ from the threshold of deterministic diffusion. As a first main point we introduced a new method for the calculation of the diffusion coefficient which allows for an extension of the results of Geisel and Nierwetberg [1] to smooth maps $f_\mu(x)$. Our central formula is the scaling law (28) or equivalently (30) for the diffusion coefficient (2), where u is the slope of $f_\mu(x)$ at the unstable fixed points, a and z describe the neighborhood of the local extrema, see (19), and $\rho(x^*)$ is the invariant density for $\epsilon=\sigma=0$ at the local extrema. As a second main issue we found that for $\epsilon>0$ and $z>1$ the deterministic diffusion coefficient is reduced in the presence of noise for noise strengths σ of order $O(\epsilon)$. For $z\rightarrow\infty$ the suppression $R(z)$ of deterministic diffusion in (34) approaches its maximal value of $\frac{1}{2}$. The comparison with numerical simulations in Fig. 5 shows very good agreement. The third main point is the discussion of our approach in view of the conceptual problems of deterministic diffusion pointed out by Fujisaka and Grossmann [3]. We derived a sufficient condition (39) for the validity of our result. Additionally, for $z>1$ the necessary and sufficient condition (38) must be fulfilled.

We end with some remarks on the diffusion coefficient for more general parameters μ but still small noise strengths σ. The main effect of the noise is to smooth the erratic behavior of the zero-noise diffusion coefficient D_0. Whereas very small periodic windows are completely wiped out by the noise, within larger windows with vanishing or infinite D_0 the diffusion coefficient becomes still small but positive or large but finite, respectively. Accordingly, there are four types of asymptotic laws [2] depending on whether one is close to the upper or lower boundary of a window with vanishing or infinite D_0. In the present investigation we treated the simplest case of an upper boundary with vanishing D_0, whereas in [24,25] the simplest case of a lower boundary with infinite D_0 was considered. We suppose that all the remaining cases can be solved in principle by similar methods, though the explicit elaboration is far from trivial. Between the boundary regions of a window described by the asymptotic laws, the diffusion coefficient is related in a simple way to an appropriately defined escape rate or phase-slip rate, as, for instance, in (9), and thus one can make use of the well-known theory of these rates [26]. In passing we note that also within a window with infinite D_0 one has noise-induced suppression of deterministic diffusion, but obviously of a rather trivial kind and without a minimum of the diffusion coefficient as a function of the noise strength.

It has to be expected that the neighborhoods of the larger windows which are not wiped out by the noise and where the asymptotic laws for the diffusion coefficient are valid do not cover the whole domain of the small windows which are wiped out by the noise. In other words, there are gaps between the larger windows where the diffusion coefficient is not known. In particular, this concerns the case $z\leq 1$ where there are often no windows apart from the trivial one below $\epsilon=0$. In contrast to all the previously considered situations, there is no kind of separation of time scales in the gaps between the larger windows. As a consequence, the diffusion coefficient can only be calculated for special examples [3,27], but no general theory is known.

In the case $z>1$ with decreasing noise strength σ asymptotic laws for the diffusion coefficient become valid for smaller and smaller windows with an increasing number of smaller and smaller gaps in between. In the limit $\sigma\rightarrow 0$ the asymptotic laws become meaningless since their validity is confined to single points, as discussed at the beginning of Sec. IV.

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[21] For simplicity we assume that u in (8), $\ln(u)$, and a in (19) are of order one. In particular, the limits $u \to \infty$ and $u \to 1$ have to be discussed separately.

[22] The explicit calculation of these distributions is very similar to the functions $w_i(x)$ in (24). For instance, if the initial distribution is given by a Gaussian of mean m_0 and variance σ_0^2 then the distribution after n time steps is given by a Gaussian of mean $u^n m_0$ and variance $\sigma^2(u^{2n}-1)/(u^2-1)+u^{2n}\sigma_0^2$ in analogy to (26).

[23] This can be made plausible as follows: Consider the sawtooth map with constant absolute slope 3 which is conjugate to the cubic Chebyshev polynomial $T_3(x)$ introduced below (36), see, e.g., [14]. For the Frobenius-Perron operator of the sawtooth map a polynomial ansatz yields a complete set of eigenfunctions with eigenvalues 9^{-i}, $i=0,1,2,\ldots$. Since the eigenvalues of the Frobenius-Perron operator are invariant under conjugation, the relaxation towards the invariant density $\rho(x)$ is governed by the eigenvalue $\frac{1}{9}$ in the case of the map $T_3(x)$. Thus a probability distribution which is already well spread out will approach $\rho(x)$ within a few time steps. From this example and in view of the property (18) it is suggestive that the same is true for all the noisy maps considered here.

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